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# INITIALISATION FOR THE 10-LEVEL MODEL

## 1. Introduction

This note contains an account of the initialisation procedure as used in the version of the program current on 1 August 1972. A general description of the program and theory is given followed by an account of the differences which exist between the procedures for the rectangular and octagonal forecast areas. The details of the algebra are given in the Appendices. The notation used in the finite difference expressions is illustrated in Fig 1 (where the grid drawn is for 'h' points of the finite difference mesh).

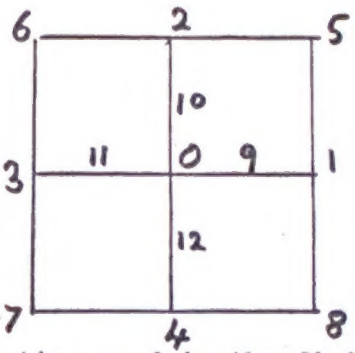
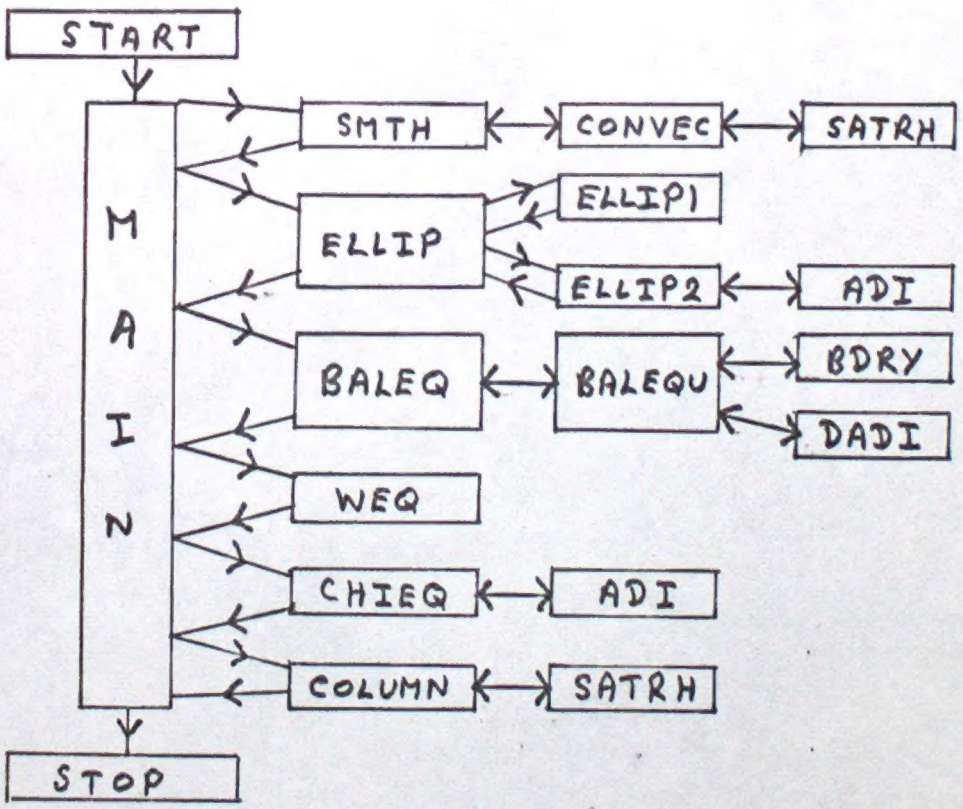



Fig 1. The numbering convention used in the finite difference expressions.

## 2. Flow diagram for the INIT program




## 3. Convective adjustment (SMTH, CONVEC, SATRH)

The lapse rate $\gamma = -\partial T/\partial z$ is calculated and compared with $\gamma_c$ where

$$\gamma_c = \begin{cases} \gamma_d & \text{for } 0 \le X < X_c \\ \gamma_d\left(\frac{1-X}{1-X_c}\right) + \gamma_s\left(\frac{X-X_c}{1-X_c}\right) & \text{for } X_c \le X \le 1 \end{cases}$$

where $\gamma_d$ is the dry adiabatic lapse rate, $\gamma_s$ is the saturated adiabatic lapse rate, X is the relative humidity and $X_c = 0.5$. If $\gamma \leq \gamma_c$ no action is taken, but if $\gamma > \gamma_c$ then the thicknesses are adjusted to give $\gamma = \gamma_c$ subject to the contraint that the total thickness between 1000 mb and 200 mb remains constant. The relative humidities of each of the layers are preserved during the adjustment.

SMTH reads down data and selects a column of thicknesses and humidities. CONVEC calculates the lapse rates, tests for instability and does the adjustment. SATRH calculates the saturated mixing ratio using a quartic polynomial for each layer as a function of the thickness of the layer.

4. Ellipticisation (ELLIP, ELLIP1, ELLIP2, ADI)

The ellipticity condition (Appendix 1 equ ( 4 )) is used in the finite difference form

$$\sum_1^4 h_i - 4h_0 + f_0^2/2g\mu_0 - [(f_1-f_3)(h_1-h_3) + (f_2-f_4)(h_2-h_4)]/4f_0$$

$$+ g[(h_1-h_3)^2 + (h_2-h_4)^2][\mu_0(\sum_1^4 \mu_i - 4\mu_0) - \tfrac{1}{4}\{(\mu_1-\mu_3)^2 + (\mu_2-\mu_4)^2\}]/8f_0^2\mu_0 - \varepsilon > 0$$

which we abbreviate to $G > 0$, $\varepsilon = 2000 \times f$ for the rectangle and $\varepsilon = 2.26718$ for the octagon (see section 9).

The program calculates G at each point and sets up an indicator field containing zeros at points where $G \geq 0$ and ones where $G < 0$. The indicator field is then adjusted to form 'pyramids', i.e. groups of ones are examined and all the interior points of the group are changed to twos; next the groups of twos are examined and all the interior points changed to threes, and so on. The next step in the process is to adjust the G field to form a new $G'$ field where $G' \geq 0$ and the mean value of $G'$ is equal to the mean value of G. This is done by starting with the highest point of the indicator pyramids and setting the corresponding G to zero, while at the same time adding $G/n$ to each of the $n$ adjacent points whose indicator fields are less than that of the point in question. The process is then repeated for the next highest point of the pyramids. The new G field (now referred to as $G'$) is again tested to see if it is positive, and if not the process is repeated.

The ellipticised h field is now calculated by solving the equation.

$$\sum_1^4 h_i - 4h_0 + f_0^2/2g\mu_0 - [(f_1-f_3)(h_1-h_3) + (f_2-f_4)(h_2-h_4)]/4f_0$$

$$+ g[(h_1-h_3)^2 + (h_2-h_4)^2][\mu_0(\sum_1^4 \mu_i - 4\mu_0) - \tfrac{1}{4}\{(\mu_1-\mu_3)^2 + (\mu_2-\mu_4)^2\}]/8f_0^2\mu_0 - \varepsilon - G' = 0$$

using alternating direction implicit methods (ADI). Finally the average difference between the ellipticised height field and the analysed height field is subtracted from the ellipticised field so as to preserve the mean height of the level.

The ellipticisation process is started at the 1000 mb level and proceeds level by level up to 100 mb. Before the procedure is begun at any level, 0.85 times the difference between the final ellipticised field and the analysed field at the level below is added to the analysed height field in question.

ELLIP selects the levels, adds on 0.85 of the height changes and prints out the average height change together with the position of the grid point where it occurs. ELLIP1 computes certain constant terms of G. ELLIP2 sets up the indicator field pyramids, derives the G' field, prints out the maximum values of the indicator field, solves for the ellipticised h field and prints out the maximum modulus change after each iteration. ADI performs a sequence of alternating direction implicit sweeps with 4 pairs of iteration parameters (using single precision arithmetic).

5. The Balance Equation (BALEQ, BALEQU, BDRY, DADI)

The first guess for the stream function $\psi$ is taken as $gh/\bar{f}$ where $\bar{f}$ is the value of f at the mid point of the region. The next guess for $\psi$ is found by solving the linear balance equation using two sets of alternating direction implicit sweeps. The linear balance equation is used in the form

$$\sum_{1}^{4}\psi_i - 4\psi_0 - 4g\left[\sum_{1}^{4}h_i - 4h_0\right]\Big/\sum_{9}^{16}f_i = 0$$

Boundary values for $\psi$ are taken as

$$\psi_0 = gh_0/f$$

$$\psi_j = \psi_{j-1} + g(h_j - h_{j-1})/f_{j-\frac{1}{2}} - \frac{1}{N}\sum_{k=1}^{N} g(h_k - h_{k-1})/f_{k-\frac{1}{2}}$$

where j and k cycle around the boundary and N is the number of boundary points. Boundary values derived in this way ensure that there is no net inflow into the region.

The full balance equation, which we may write in the form

$$\sum_{1}^{4}\psi_i^{(n)} - 4\psi_0^{(n)} + D(\psi^{(n-1)}) = 0$$

is solved in double precision arithmetic using the alternating direction implicit routine DADI.

$$D = -\left[\sum_{i=1}^{4}\psi_i^{(n-1)} + \frac{1}{2A(\psi^{(n-1)})}\left\{B(\psi^{(n-1)}) - \sqrt{B^2(\psi^{(n-1)}) - 4A(\psi^{(n-1)})C(\psi^{(n-1)})}\right\}\right]$$

and A, B and C are derived in Appendix 2. D is worked out before each entry to DADI. The term under the square root is set equal to zero if it is negative and an appropriate message printed out. The program monitors the nature of the convergence and takes the following action to accelerate convergence;

(i) if the solution is converging in an oscillatory manner (defined as four consecutive changes of sign in the largest change of $\psi$ from one iteration to the next) then the next guess for $\psi$ is taken to be $\frac{1}{2}(\psi^{(n-1)} + \psi^{(n)})$

(ii) if the solution is converging monatonically the next guess is taken as $\psi^{(n)}$

(iii) if the solution is diverging and oscillating the next guess is taken to be $\frac{1}{2}(\psi^{(n-1)} + \psi^{(n)})$ provided that it does not diverge for more than two consecutive iterations.

(iv) if the solution is diverging monatonically the program stops trying to solve the balance equation and takes the current value of $\psi$ as the best estimate.

At each iteration the program prints out the maximum modulus change and the coordinate of the point where it occurs. If the average $\frac{1}{2}(\psi^{(n-1)} + \psi^{(n)})$ is taken the message MEAN OF PSI is printed out.

BALEQ reads down data, selects a level, writes the solution for that level to a temporary data set, and arranges all the levels in column form and writes them to a permanent data set. BALEQU calculates D and solves the balance equation. BDRY works out the boundary values of $\psi$. DADI performs a set of ADI sweeps in double precision arithmetic using 4 pairs of iteration parameters.

## 6. The $\omega$-equation (WEQ)

Since the $\omega$ field is calculated at grid points displaced from the 'h' points it is necessary to adopt a new numbering convention for the grid point values in the finite difference representation for the $\omega$-equation. However we will still use the convention illustrated in fig 1, except that now we will assume that the grid drawn is for the '$\omega$' points.

The $\omega$-equation is solved using the iterative scheme

$$\omega_0^{(n)}(k) = \alpha\,\omega_0^{(n-1)}(k) + (1-\alpha)\Big[\Big\{\omega_0^{(n-1)}(k+1) + \omega_0^{(n-1)}(k-1)\Big\}K1/100^2 + \sum_{i=1}^{4}\omega_i^{(n-1)}(k) - K2\Big]\Big/\big(4 + 2K1/100^2\big)$$

where n is the iteration number, $\alpha$ is an over-relaxation parameter ( $\alpha = -1/3$), k refers to the pressure level and K1 and K2 are defined in Appendix 3.

The boundary conditions for are as follow

$\omega = 0$ at 100 mb
$\omega = 0$ around the sides
$\omega(1000) = K3\,\omega(900) + K4$ at 1000 mb

where K3 and K4 are defined in Appendix 3.

## 7. The Chi-equation (CHI EQ, ADI)

The velocity potential $\chi$ is found by solving the equation

$$\sum_{1}^{4}\chi_i - 4\chi_0 + F(\omega) = 0$$

using ADI. The notation for the grid points is the same as for the $\omega$ -equation (i.e the velocity potential is calculated at '$\omega$' points). The function $F(\omega)$ is calculated using centred differences, except at the 1000mb level one sided differences are used. Thus

$$F(\omega, 100) = \omega(200)/200\,\mu$$

$$F(\omega, 200) = \big[\omega(300) - \omega(100)\big]/200\,\mu$$

etc

$$F(\omega, 1000) = \big[\omega(1000) - \omega(900)\big]/100\,\mu$$

etc. As lateral boundary conditions for the chi-equation we use $\chi = 0$.

Having found $\chi$ the total wind field components are calculated from

$$u^* = a\left[-\psi_y + \chi_x\right]$$

$$v^* = a\left[\psi_x + \chi_y\right]$$

Around the boundaries of the region of integration (where the velocity potential is undefined) the wind is non-divergent, i.e. defined in terms of the stream function $\psi$ only.

8. Arrangement of the data into column form (COLUMN, [WMDATA, WVDATA])

The forecast program requires the data to be written as a data set on a 3330 disk in a special format (details are given in Appendix 4). COLUMN reads the individual data sets and rearranges them into the appropriate form. Zeros are placed in unused positions except for positions 28-31 in the last column of the first line which contain the time, date, month and year of the initial data as integers. The lines are written to the disk using the special I/O routines, WMDATA (for the rectangle), or WVDATA (for the octagon).

9. Differences between the octagon and rectangle

The rectangle analysis is done over a 66 x 50 region and this is reduced to a 64 x 48 region for the forecast. The octagon analysis is performed over a region 63 grid points across while the forecast area has 61 points across (both areas have 25 points in the first line). For both the rectangle and the octagon the ellipticisation and the balance equation use the larger analysis regions while the remainder of the initialisation programs (WEQ, CHIEQ) works on the forecast area (the outer boundary being 'thrown away' within the program).

In the ellipticisation stage of the program the parameter $\varepsilon$ takes different values for the rectangle and octagon. For the rectangle $\varepsilon = 2000 \times f$ where $f$ is the Coriolis parameter. An equivalent formulation for the octagon however was found to give an insufficiently large value for $\varepsilon$ in the tropics, leading to a lack of ellipticity in the height fields (the term $\nabla\psi.\nabla f$ was not well represented by $g/f\ \nabla h.\nabla f + \varepsilon$ ), and consequently a different procedure was adopted. The Coriolis parameter $f$ is taken to be constant south of latitude 18.62 °N. In the region where the coriolis parameter varied with latitude $\varepsilon$ is taken as 2.26718, while in the region where $f$ = constant $\varepsilon = 0.226718$.

APPENDIX I

## DERIVATION OF THE BALANCE EQUATION AND THE ELLIPTICITY CONDITION

The equations of motion are (neglecting friction etc.)

$$\frac{\partial u^*}{\partial t} + \mu a\left[u^* u^*_x + v^* u^*_y\right] + \omega v^*_p - f v^* + \tfrac{1}{2} a \mu_x (u^{*2} + v^{*2}) + g a h_x = 0 \quad (1)$$

$$\frac{\partial v^*}{\partial t} + \mu a\left[u^* v^*_x + v^* v^*_y\right] + \omega v^*_p + f u^* + \tfrac{1}{2} a \mu_y (u^{*2} + v^{*2}) + g a h_y = 0 \quad (2)$$

where $u^* = ua/m$ , $v^* = va/m$ , $\mu = (m/a)^2$ and $m$ is the map scale factor.

The balance equation is obtained by forming $\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2)$ and neglecting $\frac{d}{dt}(u^*_x + v^*_y)$, thus we obtain

$$a\mu_x (u^* u^*_x + v^* v^*_y) + a\mu_y (u^* v^*_x + v^* v^*_y) + a\mu (u^{*2}_x + 2 v^*_x u^*_y + v^{*2}_y)$$

$$- f(v^*_x - u^*_y) - f_x v^* + f_y u^* + \tfrac{1}{2} a (\mu_{xx} + \mu_{yy})(u^{*2} + v^{*2}) + g\mu_x (u^* u^*_x + v^* v^*_x)$$

$$+ a\mu_y (u^* u^*_y + v^* v^*_y) + g a (h_{xx} + h_{yy}) = 0$$

Put $u^* = -a\psi_y$ , $v^* = a\psi_x$

$$2a^3\mu (\psi^2_{xy} - \psi_{xx}\psi_{yy}) + a^3\mu_x (2\psi_y \psi_{xy} + \psi_x (\psi_{xx} - \psi_{yy}))$$

$$+ a^3\mu_y (2\psi_x \psi_{xy} - \psi_y (\psi_{xx} - \psi_{yy})) - af(\psi_{xx} + \psi_{yy})$$

$$- af_x \psi_x - af_y \psi_y + \tfrac{1}{2} a^3 (\mu_{xx} + \mu_{yy})(\psi^2_x + \psi^2_y) + ga(h_{xx} + h_{yy}) = 0 \quad (3)$$

The ellipticity condition is found from the relation

$$4 \frac{\partial F}{\partial \psi_{xx}} \frac{\partial F}{\partial \psi_{yy}} - \left(\frac{\partial F}{\partial \psi_{xy}}\right)^2 > 0$$

where it is assumed that the balance equation (3) has been written in the form

$$F(\psi_{xx}, \psi_{xy}, \psi_{yy}) = 0$$

Thus we obtain

$$[4a^3\mu\psi_{xy} + 2a^3(\mu_x\psi_y + \mu_y\psi_x)]^2 - 4[-2a^3\mu\psi_{yy} + a^3(\mu_x\psi_x - \mu_y\psi_y) - af]\times$$
$$\times[-2a^3\mu\psi_{xx} + a^3(\mu_y\psi_y - \mu_x\psi_x) - af] \quad < 0$$

i.e.

$$4a^4\mu^2(\psi_{xy}^2 - \psi_{xx}\psi_{yy}) + 2a^4\mu\mu_x(2\psi_y\psi_{xy} + \psi_x(\psi_{xx} - \psi_{yy}))$$
$$+ 2a^4\mu\mu_y(2\psi_x\psi_{xy} - \psi_y(\psi_{xx} - \psi_{yy})) - 2fa^2\mu(\psi_{xx} + \psi_{yy})$$
$$+ a^4(\mu_x^2 + \mu_y^2)(\psi_x^2 + \psi_y^2) - f^2 \quad < 0$$

i.e (using the balance equation (3))

$$2a^2\mu(f_x\psi_x + f_y\psi_y) - a^4\mu(\mu_{xx} + \mu_{yy})(\psi_x^2 + \psi_y^2) - 2ga^2\mu(h_{xx} + h_{yy})$$
$$+ a^4(\mu_x^2 + \mu_y^2)(\psi_x^2 + \psi_y^2) - f^2 < 0$$

i.e.

$$-2(h_{xx} + h_{yy}) - \frac{a^2}{g}(f_x\psi_x + f_y\psi_y) + \frac{a^4}{2g\mu}(\psi_x^2 + \psi_y^2)[\mu(\mu_{xx} + \mu_{yy}) - (\mu_x^2 + \mu_y^2)]$$
$$+ \frac{f^2}{2g\mu} > 0$$

Replacing $\psi$ by $gh/f$, this condition becomes (in finite differences)

$$\sum_1^4 h_i - 4h_0 + \frac{f^2}{2g\mu} - \frac{1}{4f}[(f_1 - f_2)(h_1 - h_3) + (f_2 - f_4)(h_2 - h_4)] \qquad (4)$$
$$+ \frac{g}{8f^2\mu}[(h_1 - h_3)^2 + (h_2 - h_4)^2]\left[\mu_0\left(\sum_1^4 \mu_i - 4\mu_0\right) - \frac{1}{4}\{(\mu_1 - \mu_3)^2 + (\mu_2 - \mu_4)^2\}\right] > \varepsilon$$

where $\varepsilon$ is introduced to compensate for replacing $\psi$ by $gh/f$.

APPENDIX II

## DERIVATION OF THE BALANCE EQUATION FROM THE FINITE DIFFERENCE EQUATIONS OF MOTION

The equations of motion are (neglecting friction etc.)

$$\frac{\partial u^*}{\partial t} + \mu a \left[ u^* u^*_x + v^* u^*_y \right] + \omega u^*_p - f v^* + \tfrac{1}{2} a \mu_x (u^{*2} + v^{*2}) + g a h_x = 0 \qquad (1)$$

$$\frac{\partial v^*}{\partial t} + \mu a \left[ u^* v^*_x + v^* v^*_y \right] + \omega v^*_p + f u^* + \tfrac{1}{2} a \mu_y (u^{*2} + v^{*2}) + g a h_y = 0 \qquad (2)$$

$$\mu a \left[ u^*_x + v^*_y \right] + \omega_p = 0 \qquad (3)$$

where $u^* = ua/m$, $v^* = va/m$, $\mu = (m/a)^2$ and $m$ is the map scale factor.

At $t = 0$ the variables are arranged as illustrated in Fig 1a while at $t = \frac{1}{2}\delta t$ they are arranged as in Fig 1b. Thus at $t = 0$ $\partial u^*/\partial t$ is evaluated at the 'v' points of Fig 1a and $\partial v^*/\partial t$ is evaluated at the 'u' points. It is clear therefore that $\partial(u^*_x + v^*_y)/\partial t$ must be evaluated at the 'h' points of the finite difference mesh at $t = 0$.

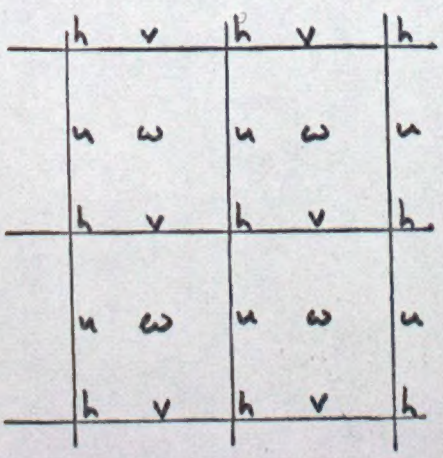



Fig 1a.

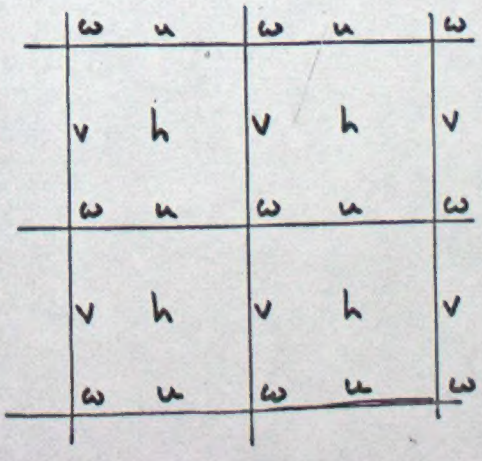



Fig 1b.

In order to derive the balance equation we use equations (1) and (2) to form the equation

$$a\left[\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2)\right] = 0 \qquad (4)$$

and then set

$$u^* = -a\,\psi_y$$

$$v^* = a\,\psi_x$$

All the differential operations are done in finite differences and the finite difference representations of (1) and (2) must be those which are used to produce the subsequent forecast.

In the following algebra the coordinate system is as illustrated in fig 2, with the central 'h' point designated (0,0) and the neighbouring 'h' points being unit distance away (in equations (1) and (2) 'a' represents the actual distance between the 'h' points at the north pole).

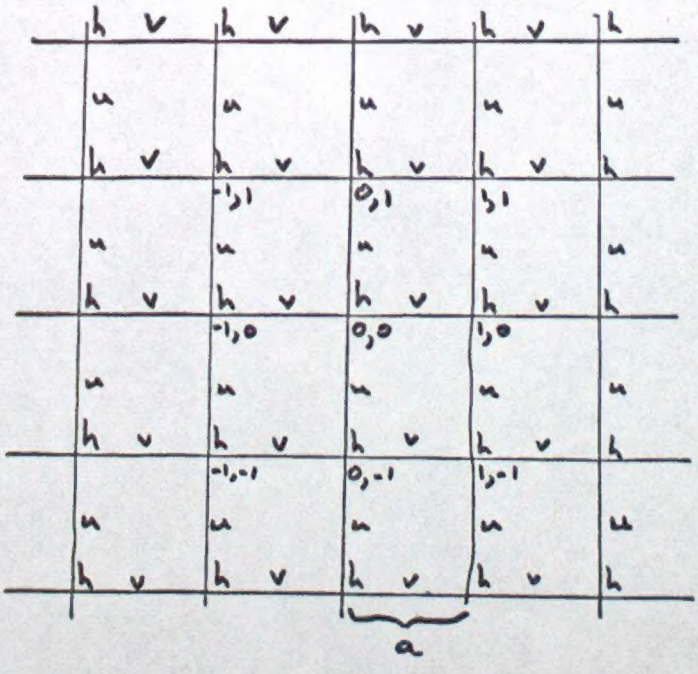



Fig 2

We will consider each of the terms of equation (4) separately (in the interests of simplicity we will drop the stars in the subsequent algebra).

1) $a^2 \frac{\partial}{\partial x}\left[\mu u^* u^*_x\right]$

$$= \frac{1}{8}\left(u_{1,\frac{1}{2}} + u_{1,-\frac{1}{2}} + u_{0,\frac{1}{2}} + u_{0,-\frac{1}{2}}\right)\left(u_{1,\frac{1}{2}} + u_{1,-\frac{1}{2}} - u_{0,\frac{1}{2}} - u_{0,-\frac{1}{2}}\right)\mu_{\frac{1}{2},0}$$

$$- \frac{1}{8}\left(u_{0,\frac{1}{2}} + u_{0,-\frac{1}{2}} + u_{-1,\frac{1}{2}} + u_{-1,-\frac{1}{2}}\right)\left(u_{0,\frac{1}{2}} + u_{0,-\frac{1}{2}} - u_{-1,\frac{1}{2}} - u_{-1,-\frac{1}{2}}\right)\mu_{-\frac{1}{2},0}$$

$$= \frac{1}{8}\Big[\left\{\left(u_{1,\frac{1}{2}} + u_{1,-\frac{1}{2}}\right)^2 - \left(u_{0,\frac{1}{2}} + u_{0,-\frac{1}{2}}\right)^2\right\}\mu_{\frac{1}{2},0}$$

$$- \left\{\left(u_{0,\frac{1}{2}} + u_{0,-\frac{1}{2}}\right)^2 - \left(u_{-1,\frac{1}{2}} + u_{-1,-\frac{1}{2}}\right)^2\right\}\mu_{-\frac{1}{2},0}\Big]$$

$$= \frac{1}{8}\left[\mu_{\frac{1}{2},0}\left(\psi_{1,-1} - \psi_{1,1}\right)^2 - \left(\mu_{\frac{1}{2},0} + \mu_{-\frac{1}{2},0}\right)\left(\psi_{0,-1} - \psi_{0,1}\right)^2 + \mu_{-\frac{1}{2},0}\left(\psi_{-1,-1} - \psi_{-1,1}\right)^2\right]$$

2) $a^2 \frac{\partial}{\partial x}\left[\mu v^* u^*_y\right]$

$$= \frac{1}{2}\left[\mu_{\frac{1}{2},0}\, v_{\frac{1}{2},0}\left(u_{1,\frac{1}{2}} + u_{0,\frac{1}{2}} - u_{1,-\frac{1}{2}} - u_{0,-\frac{1}{2}}\right) - \mu_{-\frac{1}{2},0}\, v_{-\frac{1}{2},0}\left(u_{0,\frac{1}{2}} + u_{-1,\frac{1}{2}} - u_{0,-\frac{1}{2}} - u_{-1,-\frac{1}{2}}\right)\right]$$

$$= \frac{1}{2}\Big[\mu_{\frac{1}{2},0}\left(\psi_{1,0} - \psi_{0,0}\right)\left(2\psi_{1,0} - \psi_{1,1} + 2\psi_{0,0} - \psi_{0,1} - \psi_{1,-1} - \psi_{0,-1}\right)$$

$$- \mu_{-\frac{1}{2},0}\left(\psi_{0,0} - \psi_{-1,0}\right)\left(2\psi_{0,0} - \psi_{0,1} + 2\psi_{-1,0} - \psi_{-1,1} - \psi_{0,-1} - \psi_{-1,-1}\right)\Big]$$

3) $a^2 \frac{\partial}{\partial y}\left[\mu u^* v^*_x\right]$

$$= \frac{1}{2}\left[\mu_{0,\frac{1}{2}}\, u_{0,\frac{1}{2}}\left(v_{\frac{1}{2},0} + v_{\frac{1}{2},1} - v_{-\frac{1}{2},0} - v_{-\frac{1}{2},1}\right) - \mu_{0,-\frac{1}{2}}\, u_{0,-\frac{1}{2}}\left(v_{\frac{1}{2},0} + v_{\frac{1}{2},-1} - v_{-\frac{1}{2},0} - v_{-\frac{1}{2},-1}\right)\right]$$

$$= \frac{1}{2}\Big[\mu_{0,\frac{1}{2}}\left(\psi_{0,0} - \psi_{0,1}\right)\left(\psi_{1,0} - 2\psi_{0,0} + \psi_{1,1} - 2\psi_{0,1} + \psi_{-1,0} + \psi_{-1,1}\right)$$

$$- \mu_{0,-\frac{1}{2}}\left(\psi_{0,-1} - \psi_{0,0}\right)\left(\psi_{1,0} - 2\psi_{0,0} + \psi_{1,-1} - 2\psi_{0,-1} + \psi_{-1,0} + \psi_{-1,-1}\right)$$

4) $$a^2 \frac{\partial}{\partial y}\left[\mu v^* v_y^*\right]$$

$$= \frac{1}{8}\left[\mu_{0,\frac{1}{2}}(\psi_{1,1}-\psi_{-1,1})^2 - (\mu_{0,\frac{1}{2}}+\mu_{0,-\frac{1}{2}})(\psi_{1,0}-\psi_{-1,0})^2 + \mu_{0,-\frac{1}{2}}(\psi_{1,-1}-\psi_{-1,-1})^2\right]$$

5) $$a\frac{\partial}{\partial x}\left[\omega u_p^*\right] + a\frac{\partial}{\partial y}\left[\omega v_p^*\right]$$

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6) $$a\frac{\partial}{\partial x}\left[-f v^*\right] + a\frac{\partial}{\partial y}\left[f u^*\right]$$

$$= f_{-\frac{1}{2},0} v_{-\frac{1}{2},0} - f_{\frac{1}{2},0} v_{\frac{1}{2},0} + f_{0,\frac{1}{2}} u_{0,\frac{1}{2}} - f_{0,-\frac{1}{2}} u_{0,-\frac{1}{2}}$$

$$= \psi_{0,0}\left(f_{-\frac{1}{2},0} + f_{\frac{1}{2},0} + f_{0,\frac{1}{2}} + f_{0,-\frac{1}{2}}\right) - f_{-\frac{1}{2},0}\psi_{-1,0} - f_{\frac{1}{2},0}\psi_{1,0} - f_{0,\frac{1}{2}}\psi_{0,1} - f_{0,-\frac{1}{2}}\psi_{0,-1}$$

7) $$a^2 \frac{\partial}{\partial x}\left[\frac{1}{2}\mu_{x,c}\left(u^{*2}+v^{*2}\right)\right]$$

$$= \frac{1}{2}\left[\left\{\frac{1}{16}\left(u_{0,\frac{1}{2}}+u_{1,\frac{1}{2}}+u_{0,-\frac{1}{2}}+u_{1,-\frac{1}{2}}\right)^2 + v_{\frac{1}{2},0}^2\right\}(\mu_{1,0}-\mu_{0,0})\right.$$
$$\left. - \left\{\frac{1}{16}\left(u_{0,\frac{1}{2}}+u_{-1,\frac{1}{2}}+u_{0,-\frac{1}{2}}+u_{-1,-\frac{1}{2}}\right)^2 + v_{-\frac{1}{2},0}^2\right\}(\mu_{0,0}-\mu_{-1,0})\right]$$

$$= \frac{1}{2}\left[\left\{\frac{1}{16}\left(\psi_{0,-1}-\psi_{0,1}+\psi_{1,-1}-\psi_{1,1}\right)^2 + \left(\psi_{1,0}-\psi_{0,0}\right)^2\right\}(\mu_{1,0}-\mu_{0,0})\right.$$
$$\left. - \left\{\frac{1}{16}\left(\psi_{-1,-1}-\psi_{-1,1}+\psi_{0,-1}-\psi_{0,1}\right)^2 + \left(\psi_{0,0}-\psi_{-1,0}\right)^2\right\}(\mu_{0,0}-\mu_{-1,0})\right]$$

$$= \frac{1}{2}\psi_{0,0}^2\left(\mu_{1,0}+\mu_{-1,0}-2\mu_{0,0}\right) - \psi_{0,0}\left[\psi_{1,0}(\mu_{1,0}-\mu_{0,0}) + \psi_{-1,0}(\mu_{-1,0}-\mu_{0,0})\right]$$
$$+ \frac{1}{32}\left[\left\{\left(\psi_{0,-1}-\psi_{0,1}+\psi_{1,-1}-\psi_{1,1}\right)^2 + 16\,\psi_{1,0}^2\right\}(\mu_{1,0}-\mu_{0,0})\right.$$
$$\left. + \left\{\left(\psi_{-1,-1}-\psi_{-1,1}+\psi_{0,-1}-\psi_{0,1}\right)^2 + 16\,\psi_{-1,0}^2\right\}(\mu_{-1,0}-\mu_{0,0})\right]$$

8) $$a^2 \frac{\partial}{\partial y}\left[\tfrac{1}{2}\mu_y\left(u^{*2}+v^{*2}\right)\right]$$

$$= \tfrac{1}{2}\Psi_{0,0}^2\left(\mu_{0,1}+\mu_{0,-1}-2\mu_{0,0}\right) - \Psi_{0,0}\left[\Psi_{0,1}\left(\mu_{0,1}-\mu_{0,0}\right)+\Psi_{0,-1}\left(\mu_{0,-1}-\mu_{0,0}\right)\right]$$

$$+\frac{1}{32}\Big[\left\{\left(\Psi_{-1,0}-\Psi_{1,0}+\Psi_{-1,1}-\Psi_{1,1}\right)^2+16\,\Psi_{0,1}^2\right\}\left(\mu_{0,1}-\mu_{0,0}\right)$$

$$+\left\{\left(\Psi_{-1,-1}-\Psi_{1,-1}+\Psi_{-1,0}-\Psi_{1,0}\right)^2+16\,\Psi_{0,-1}^2\right\}\left(\mu_{0,-1}-\mu_{0,0}\right)\Big]$$

9) $$a^2 g\left(h_{xx}+h_{yy}\right)$$

$$= g\left(h_{1,0}+h_{0,1}+h_{-1,0}+h_{0,-1}-4h_{0,0}\right)$$

Adding these terms together we get

$$\hat{A}\,\Psi_{0,0}^2 + \hat{B}\,\Psi_{0,0} + \hat{C} = 0$$

where

$$\hat{A} = -\left(\mu_{\frac{1}{2},0}+\mu_{-\frac{1}{2},0}+\mu_{0,\frac{1}{2}}+\mu_{0,-\frac{1}{2}}\right)+\tfrac{1}{2}\left(\mu_{1,0}+\mu_{-1,0}+\mu_{0,1}+\mu_{0,-1}-4\mu_{0,0}\right)$$

$$\hat{B} = \tfrac{1}{2}\Big[\mu_{\frac{1}{2},0}\left(\Psi_{1,1}+\Psi_{0,1}+\Psi_{1,-1}+\Psi_{0,-1}\right)$$

$$+\mu_{-\frac{1}{2},0}\left(\Psi_{0,1}+\Psi_{-1,1}+\Psi_{0,-1}+\Psi_{-1,-1}\right)$$

$$+\mu_{0,\frac{1}{2}}\left(\Psi_{1,0}+\Psi_{1,1}+\Psi_{-1,0}+\Psi_{-1,1}\right)$$

$$+\mu_{0,-\frac{1}{2}}\left(\Psi_{1,0}+\Psi_{1,-1}+\Psi_{-1,0}+\Psi_{-1,-1}\right)\Big]$$

$$+\left(f_{-\frac{1}{2},0}+f_{\frac{1}{2},0}+f_{0,\frac{1}{2}}+f_{0,-\frac{1}{2}}\right)$$

$$-\left[\Psi_{1,0}\left(\mu_{1,0}-\mu_{0,0}\right)+\Psi_{-1,0}\left(\mu_{-1,0}-\mu_{0,0}\right)+\Psi_{0,1}\left(\mu_{0,1}-\mu_{0,0}\right)+\Psi_{0,-1}\left(\mu_{0,-1}-\mu_{0,0}\right)\right]$$

$$\hat{C} = \frac{1}{8}\Big[\mu_{\frac{1}{2},0}(\psi_{1,-1}-\psi_{1,1})^2-(\mu_{\frac{1}{2},0}+\mu_{-\frac{1}{2},0})(\psi_{0,-1}-\psi_{0,1})^2+\mu_{-\frac{1}{2},0}(\psi_{-1,-1}-\psi_{-1,1})^2$$
$$+\mu_{0,\frac{1}{2}}(\psi_{1,1}-\psi_{-1,1})^2-(\mu_{0,\frac{1}{2}}+\mu_{0,-\frac{1}{2}})(\psi_{1,0}-\psi_{-1,0})^2+\mu_{0,-\frac{1}{2}}(\psi_{1,-1}-\psi_{-1,-1})^2$$
$$+4\mu_{\frac{1}{2},0}\psi_{1,0}(2\psi_{1,0}-\psi_{1,1}-\psi_{0,1}-\psi_{1,-1}-\psi_{0,-1})$$
$$+4\mu_{-\frac{1}{2},0}\psi_{-1,0}(2\psi_{-1,0}-\psi_{0,1}-\psi_{-1,1}-\psi_{0,-1}-\psi_{-1,-1})$$
$$+4\mu_{0,\frac{1}{2}}\psi_{0,1}(2\psi_{0,1}-\psi_{1,0}-\psi_{1,1}-\psi_{-1,0}-\psi_{-1,1})$$
$$+4\mu_{0,-\frac{1}{2}}\psi_{0,-1}(2\psi_{0,-1}-\psi_{1,0}-\psi_{1,-1}-\psi_{-1,0}-\psi_{-1,-1})\Big]$$
$$-\Big[f_{-\frac{1}{2},0}\psi_{-1,0}+f_{\frac{1}{2},0}\psi_{1,0}+f_{0,\frac{1}{2}}\psi_{0,1}+f_{0,-\frac{1}{2}}\psi_{0,-1}\Big]$$
$$+g\big[h_{1,0}+h_{0,1}+h_{-1,0}+h_{0,-1}-4h_{0,0}\big]$$
$$+\frac{1}{32}\Big[\big\{(\psi_{0,-1}-\psi_{0,1}+\psi_{1,-1}-\psi_{1,1})^2+16\psi_{1,0}^2\big\}(\mu_{1,0}-\mu_{0,0})$$
$$+\big\{(\psi_{-1,-1}-\psi_{-1,1}+\psi_{0,-1}-\psi_{0,1})^2+16\psi_{-1,0}^2\big\}(\mu_{-1,0}-\mu_{0,0})$$
$$+\big\{(\psi_{-1,0}-\psi_{1,0}+\psi_{-1,1}-\psi_{1,1})^2+16\psi_{0,1}^2\big\}(\mu_{0,1}-\mu_{0,0})$$
$$+\big\{(\psi_{-1,-1}-\psi_{1,-1}+\psi_{-1,0}-\psi_{1,0})^2+16\psi_{0,-1}^2\big\}(\mu_{0,-1}-\mu_{0,0})\Big]$$

Now defining $A = \hat{A}/4$, $B = \hat{B}$, $C = 4\hat{C}$ the balance equation may be written

$$\sum_1^4 \psi_i - 4\psi_0 + D = 0$$

where $$D \equiv -\sum_1^4 \psi_i + \frac{1}{2A}\left[-B \pm \sqrt{B^2-4AC}\right]$$

(the positive sign being taken in the southern hemisphere).

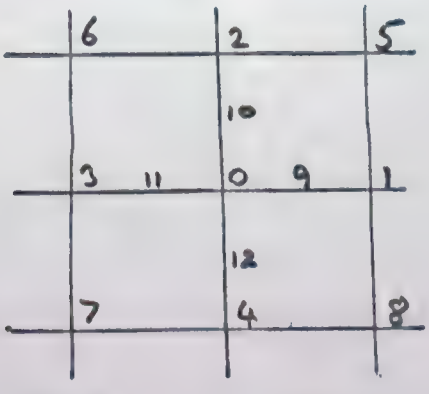


Fig 3

Using the notation of fig 3

$$A = -\frac{1}{4}\sum_{9}^{12}\mu_i + \frac{1}{8}\left(\sum_{1}^{4}\mu_i - 4\mu_0\right)$$

$$\begin{aligned} B = \frac{1}{2}\Big[ & \mu_9(\psi_5+\psi_2+\psi_4+\psi_8) + \mu_{11}(\psi_2+\psi_6+\psi_7+\psi_4) \\ & + \mu_{10}(\psi_5+\psi_6+\psi_3+\psi_1) + \mu_{12}(\psi_1+\psi_3+\psi_7+\psi_8)\Big] \\ & + \sum_{9}^{12} f_i + \sum_{1}^{4}\psi_i(\mu_0-\mu_i) \end{aligned}$$

$$\begin{aligned} C = \frac{1}{2}\Big[ & \mu_9(\psi_8-\psi_5)^2 - (\mu_9+\mu_{11})(\psi_4-\psi_2)^2 + \mu_{11}(\psi_7-\psi_6)^2 \\ & + \mu_{10}(\psi_5-\psi_6)^2 - (\mu_{10}+\mu_{12})(\psi_1-\psi_3)^2 + \mu_{12}(\psi_8-\psi_7)^2 \\ & + 4\mu_9\psi_1(2\psi_1-\psi_5-\psi_2-\psi_8-\psi_4) + 4\mu_{11}\psi_3(2\psi_3-\psi_2-\psi_6-\psi_4-\psi_7) \\ & + 4\mu_{10}\psi_2(2\psi_2-\psi_1-\psi_5-\psi_3-\psi_6) + 4\mu_{12}\psi_4(2\psi_4-\psi_1-\psi_8-\psi_3-\psi_7)\Big] \\ & - 4\left[f_{11}\psi_3 + f_9\psi_1 + f_{10}\psi_2 + f_{12}\psi_4\right] + 4g\left[\sum_{1}^{4}h_i - 4h_0\right] \\ & + \frac{1}{8}\Big[\left\{(\psi_4-\psi_2+\psi_8-\psi_5)^2 + 16\psi_1^2\right\}(\mu_1-\mu_0) \\ & + \left\{(\psi_7-\psi_6+\psi_4-\psi_2)^2 + 16\psi_3^2\right\}(\mu_3-\mu_0) \\ & + \left\{(\psi_3-\psi_1+\psi_6-\psi_5) + 16\psi_2^2\right\}(\mu_2-\mu_0) \\ & + \left\{(\psi_7-\psi_8+\psi_3-\psi_1) + 16\psi_4^2\right\}(\mu_4-\mu_0)\Big] \end{aligned}$$

APPENDIX III

a) THE DERIVATION OF THE ω-EQUATION

We use the equations of motion and thermodynamics in the form

$$\frac{\partial u^*}{\partial t} + \mu a\left[u^* u^*_x + v^* u^*_y\right] + \omega u^*_p - fv^* + \tfrac{1}{2} a \mu_x (u^{*2} + v^{*2}) + g a h_x = 0 \quad (1)$$

$$\frac{\partial v^*}{\partial t} + \mu a\left[u^* v^*_x + v^* v^*_y\right] + \omega v^*_p + fu^* + \tfrac{1}{2} a \mu_y (u^{*2} + v^{*2}) + g a h_y = 0 \quad (2)$$

$$\frac{\partial}{\partial t} h_p + \mu a\left[u^* h_{px} + v^* h_{py}\right] + \frac{\sigma}{g}\omega \quad (3)$$

together with the continuity equation

$$\mu a\left[u^*_x + v^*_y\right] + \omega_p = 0 \quad (4)$$

where $u^* = au/m$, $v^* = av/m$, $\mu = (m/a)^2$ and we approximate the stability parameter $\sigma$ by a function of $p$ only calculated from the ICAO standard atmosphere.

From (1), (2) and (4) we derive a vorticity equation which may be written

$$\frac{\partial}{\partial t}(a\mu\zeta^*) + \mu a\left[u^* \frac{\partial}{\partial x}(\mu a \zeta^* + f) + v^* \frac{\partial}{\partial y}(\mu a \zeta^* + f)\right] - (\mu a \zeta^* + f)\frac{\partial \omega}{\partial p}$$

$$+ \omega \frac{\partial}{\partial p}(\mu a \zeta^*) + \mu a \underline{k} \cdot \nabla\omega \times \frac{\partial \underline{v}^*}{\partial p} = 0 \quad (5)$$

where $\zeta^* = v^*_x - u^*_y$ and $\underline{k}$ is a unit vector in the vertical direction. The ω-equation is derived from the following relationship (see Scientific Paper No 22)

$$\frac{\partial}{\partial t}\left(f \frac{\partial \zeta^*}{\partial p}\right) - \frac{\partial}{\partial t}\left(g a \nabla^2 \frac{\partial h}{\partial p}\right) = 0 \quad (6)$$

Substituting from (3) and (5) into (6) we arrive at the following equation (after a minor approximation)

$$\sigma\mu a^2\nabla^2\omega + f(\mu a\zeta^* + f)\frac{\partial^2\omega}{\partial p^2} = \mu a f\frac{\partial}{\partial p}\left\{\left[u^*\frac{\partial}{\partial x} + v^*\frac{\partial}{\partial y}\right](\mu a\zeta^* + f)\right\}$$
$$- g\mu a^2\nabla^2\left\{\mu a\left[u^* h_{px} + v^* h_{py}\right]\right\}$$

This may now be written in finite differences

$$\sum_1^4 \omega_i^{(n)}(k) + \left(\omega_0^{(n)}(k+1) + \omega_0^{(n)}(k-1)\right) K1/100^2 - \left(4 + 2K1/100^2\right)\omega_0^{(n)}(k) - K2 = R^{(n)}$$

where $R^{(n)}$ is the residual and

$$K1 = f(\mu a\zeta^* + f)/\sigma\mu$$

$$K2 = \left\{af\frac{\partial}{\partial p}\left\{\left[u^*\frac{\partial}{\partial x} + v^*\frac{\partial}{\partial y}\right](\mu a\zeta^* + f) - ga^2\nabla^2\left\{\mu a\left[u^* h_{px} + v^* h_{py}\right]\right\}\right\}\right\}/\sigma$$

Using over-relaxation this becomes

$$\sum_1^4 \omega_i^{(n)}(k) + \left(\omega_0^{(n)}(k+1) + \omega_0^{(n)}(k-1)\right) K1/100^2 - \left(4 + 2K1/100^2\right)\omega_0^{(n+1)}(k) - K2 = \alpha R^{(n)}$$

i.e.

$$\omega_0^{(n+1)}(k) = \alpha\,\omega_0^{(n)}(k) + (1-\alpha)\left[\left\{\omega_0^{(n)}(k+1) + \omega_0^{(n)}(k-1)\right\} K1/100^2 + \sum_1^4 \omega_i^{(n)}(k) - K2\right]/\left(4 + 2K1/100^2\right)$$

where $\alpha$ is an over-relaxation parameter.

b) THE LOWER BOUNDARY CONDITION

The tendency equation has the form

$$\frac{\partial h_{1000}}{\partial t} + \mu a\left[u^*\frac{\partial}{\partial x} + v^*\frac{\partial}{\partial y}\right](h - H) + \omega\frac{\partial h}{\partial p} = 0 \qquad (7)$$

where $H(x,y)$ is the height of the topography.

Defining

$$u^* = u_1^* + u_2^*$$

$$v^* = v_1^* + v_2^*$$

where

$$u_1^* = -a\psi_y \quad , \quad u_2^* = a\chi_x$$

$$v_1^* = a\psi_x \quad , \quad v_2^* = a\chi_y$$

equation (7) may be written after some approximations (see Scientific Paper No 22)

$$h\frac{\partial \omega}{\partial p} + \mu a\left[u_1^*\frac{\partial}{\partial x} + v_1^*\frac{\partial}{\partial y}\right](h-H) + \omega\frac{\partial h}{\partial p} = 0$$

In finite differences this becomes

$$\omega(1000) = K3\,\omega(900) + K4$$

where

$$K3 = \frac{h(1000)}{2h(1000) - h(900)}$$

$$K4 = \frac{-100\mu a\left[u_1^*\frac{\partial}{\partial x} + v_1^*\frac{\partial}{\partial y}\right](h-H)}{2h(1000) - h(900)}$$

APPENDIX IV

## THE ARRANGEMENT of DATA IN COLUMN FORM FOR THE FORECAST PROGRAMS

N.B. Items marked with an asterisk are set equal to zero by the initialisation program.

$.^{h}_{1}$ $.^{v}_{2}$ $.^{u}_{3}$ $.^{w}_{4}$ The number of time steps*, $2 \times \delta t$*, and initial TT/DD/MM/YY are stored as integers in words 26-31 of the last column of the first line.

| FORTRAN | ASSEMBLER | ITEM | UNITS |
|---|---|---|---|
| 1 | $\phi - 3$ | H topography (stored negatively) | m |
| 2-1$\phi$ | 4-39 | h' thickness less ICAO 150-950 mb | m |
| 11 | 4$\phi$-43 | $h_{10}$ 1000 mb height less ICAO | m |
| 12-21 | 44-83 | ua/m 100-1000 mb | $m^2 s^{-1}$ |
| 22-31 | 84-123 | va/m 100-1000 mb | $m^2 s^{-1}$ |
| 32-41 | 124-163 | $\omega$ 100-1000 mb | $mb\ s^{-1}$ |
| 42,43 | 164-171 | $f_2$, $f_3$ | $s^{-1}$ |
| 44-5$\phi$ | 172-199 | r humidity mixing ratio 350-950 mb | $gm.kg^{-1}$ |
| 51-54 | 200-215 | $(m/a)^2$ 1, 2, 3, 4 | $m^{-2}$ |
| 55 | 216-219 | To sea surface temperature (sea) | $^{o}k$ |
| | | $\lambda$ longitude (land) | o |
| 56 | 220-223 | ro sea surface humidity (sea) | $gm.kg^{-1}$ |
| | | $\sin\phi$ sine latitude (land) | |
| 57* | 224-227 | $\delta h'$ 900-1000 mb thickness change (per double time step due to surface fluxes) | m |
| 58* | 228-231 | $\delta r$ 95$\phi$ mb humidity change (per double time step due to surface fluxes) | $g\ kg^{-1}$ |
| 59,6$\phi$* | 232-239 | Acc. dynamic, acc. convective rainfall | mm |
| 61,62* | 24$\phi$-247 | Rate of dynamic, rate of convective rainfall | $mm\ hr^{-1}$ |
| 63 | 248 | Precipitation type | |
| | 249 | Convective record | |
| | 250 | Diffusion indicator | |
| | 251 | Climatic type / climatic snow cover index | |
| 64-67* | 252-267 | Temperature changes per double time step due to long wave cooling 100-300, 300-500, 500-800, 800-1000 mb | $^{o}k$ |
| 68-7$\phi$* | 268-295 | not used | |
| 75* | 296-299 | 9-33 hr accumulated rain (rectangle $\phi\phi$2 only) | mm |
| 76-8$\phi$* | 3$\phi\phi$ 319 | 1000 mb heights at hourly intervals during previous 6 hrs (rectangle only) | m |